Theorem (Zeros of sine and cosine) The zeros of $\sin z / \cos z$ are precisely the zeros of the sine and cosine functions of a real variable:

$$\sin z = 0 \quad \text{if and only if} \quad z = k\pi, \ k \in \mathbb{Z}$$

$$\cos z = 0 \quad \text{if and only if} \quad z = k\pi + \pi/2, \ k \in \mathbb{Z}.$$

Proof. Assume $z = k\pi$. Then $\sin z = \sin k\pi = 0$ since $k\pi \in \mathbb{R}$. Similarly, if $z = k\pi + \pi/2$, then $\cos z = \cos k\pi + \pi/2 = 0$.

Conversely, assume $\sin z = 0$. Then

$$0 = |\sin z|^2 = \sin^2 x + \sinh^2 y.$$

Hence, $\sin x = 0$ and $\sinh y = 0$. Hence, $x = k\pi$ and $y = 0$. So $z = k\pi$ as claimed. Now, assume $\cos z = 0$. Then

$$0 = \cos z \overset{(11)}{=} -\sin(z - \pi/2).$$

Hence, $z - \pi/2 = k\pi$. ∎

Definition (tangent, cotangent, secant, cosecant) The tangent, cotangent, secant, and cosecant functions are defined in terms of sine and cosine:

$$\tan z \overset{\text{def}}{=} \frac{\sin z}{\cos z}, \ z \neq k\pi + \pi/2 \qquad \sec z \overset{\text{def}}{=} \frac{1}{\cos z}, \ z \neq k\pi + \pi/2$$

$$\cot z \overset{\text{def}}{=} \frac{\cos z}{\sin z}, \ z \neq k\pi \qquad \csc z \overset{\text{def}}{=} \frac{1}{\sin z}, \ z \neq k\pi$$

//

All of these function are analytic on the stated domain since $\sin z, \cos z$ are. Also, they all reduce to the ordinary trig functions when z is real, since sine and cosine do. The derivatives are exactly as expected. //

Hyperbolic Trig Functions

The complex exponential function can be decomposed as a sum of an even and an odd function:

$$e^z = \frac{e^z + e^{-z}}{2} + \frac{e^z - e^{-z}}{2}$$

We define the hyperbolic cosine and sine functions of a complex variable to be the even and odd part of e^z, respectively:

$$\underline{\cosh} \, z \overset{\text{def}}{=} \frac{e^z + e^{-z}}{2} \qquad \underline{\sinh} \, z \overset{\text{def}}{=} \frac{e^z - e^{-z}}{2}.$$

These functions are entire since e^z and e^{-z} are. and

$$\frac{d}{dz} \sinh z = \cosh z \qquad \frac{d}{dz} \cosh z = \sinh z.$$

They also reduce to the ordinary hyperbolic functions when $z = x \in \mathbb{R}$. //

Proposition (Relation to sine/cosine)

(1) $-i \sinh iz = \sin z$ (3) $\cosh iz = \cos z$

(2) $-i \sin iz = \sinh z$ (4) $\cos iz = \cosh z$.

Proof.

(1)
$$-i \sinh iz = -i\left(\frac{e^{iz} - e^{-iz}}{2}\right) = \frac{e^{iz} - e^{-iz}}{2i} = \sin z.$$

The others are similar. ∎

Corollary (Hyperbolic functions are periodic) The functions $\sinh z$ and $\cosh z$ have a period of $2\pi i$.

Proof. To prove this, we need to show $\sinh z + 2\pi i = \sinh z$.
We have

$$\sinh(z + 2\pi i) \overset{(2)}{=} -i \sin(i(z+2\pi i))$$
$$= -i \sin(iz - 2\pi)$$
$$= -i \sin iz$$
$$\overset{(2)}{=} \sinh z.$$

The proof for cosh is similar. ∎

Proposition (Various Identities)

(1) $\sinh -z = -\sinh z$

(2) $\cosh -z = \cosh z$

(3) $\cosh^2 z - \sin^2 z = 1$

(4) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

(5) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(6) $\sinh z = \sinh x \cos y + i \cosh x \sin y$

(7) $\cosh z = \cosh x \cos y + i \sinh x \sin y$

(8) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

(9) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

Proof. All can be proved by applying preceding prop. and using ordinary trig identities. To prove (3), start with

$$\sin^2 iz + \cos^2 iz = 1.$$

Then by (2) and (4) of prop.,

$$(-i \sinh z)^2 + \cosh^2 z = 1.$$

Hence,

$$\cosh^2 z - \sinh^2 z = 1.$$ ∎

Theorem (zeros of sinh/cosh) The zeros of $\sinh z$ and $\cosh z$ all lie on the imaginary axis. Precisely,

(a) $\sinh z = 0 \iff z = k\pi i, \ k \in \mathbb{Z}$

(b) $\cosh z = 0 \iff z = (\pi/2 + k\pi) i, \ k \in \mathbb{Z}$.

Proof. (of (a))

$$\begin{aligned} \sinh z = 0 &\overset{(2)}{\iff} -i \sin iz = 0 \\ &\iff \sin iz = 0 \\ &\iff iz = k\pi, \ k \in \mathbb{Z} \\ &\iff z = -k\pi i, \ k \in \mathbb{Z} \\ &\iff z = k\pi i, \ k \in \mathbb{Z}. \end{aligned}$$

Proof for $\cosh z$: write $\cosh z$ in terms of $\sinh z$ and apply (a).

■

Now that we know the zeros, we can define the hyperbolic tangent:

$$\tanh z \overset{\text{def}}{=} \frac{\sinh z}{\cosh z}, \quad z \neq (\pi/2 + k\pi) i.$$

The rest of the hyperbolic functions are the reciprocals of sinh, cosh, tanh and are defined on the domains specified by the preceding theorem. All are analytic on their domain of definition and the derivatives are as expected.

//

Inverse Trig Functions

To find an inverse of $\sin z$, we write $w = \sin^{-1} z$ and try to solve the equation for w:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Since $\sin z$ is not one-to-one, the best we can hope for is a multiple-valued function. To solve this equation, multiply through by $2ie^{iw}$:

$$(e^{iw})^2 - 2ize^{iw} - 1 = 0.$$

By quadratic formula (PSet 1 - P1)

$$e^{iw} = \frac{2iz + (-4z^2+4)^{1/2}}{2}$$

$$= iz + (1-z^2)^{1/2}$$

Hence, taking logarithms..

$$\boxed{\sin^{-1} z = w = -i \log(iz + (1-z^2)^{1/2})}$$

//

Similar computations for $\cos z$ and $\tan z$ produce:

$$\cos^{-1} z = -i \log(z + i(1-z^2)^{1/2})$$

$$\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}$$

A branch of any of these is determined by specifying a branch of the logarithm and a branch of the square root. In that case, the functions are analytic by the chain rule and

$$\boxed{\begin{aligned} \frac{d}{dz}\sin^{-1} z &= \frac{1}{(1-z^2)^{1/2}} \\ \frac{d}{dz}\cos^{-1} z &= -\frac{1}{(1-z^2)^{1/2}} \\ \frac{d}{dz}\tan^{-1} z &= \frac{1}{1+z^2} \end{aligned}}$$

Lets verify the third one:

$$\frac{d}{dz}\tan^{-1} z = \frac{d}{dz}\frac{i}{2}\log\frac{i+z}{i-z}$$

$$= \frac{i}{2}\frac{d}{dz}\log\frac{i+z}{i-z}$$

$$= \frac{i}{2}\left(\frac{i-z}{i+z}\right)\cdot\frac{d}{dz}\left(\frac{i+z}{i-z}\right)$$

$$= \frac{i}{2}\left(\frac{i-z}{i+z}\right)\left(\frac{1(i-z)-(-1)(i+z)}{(i-z)^2}\right)$$

$$= \frac{i}{2}\left(\frac{i-z}{i+z}\right)\frac{2i}{(i-z)^2}$$

$$= (-1)\left(\frac{1}{(i+z)(i-z)}\right) = (-1)\left(\frac{1}{-1-z^2}\right) = \frac{1}{1+z^2}$$

//

Inverse hyperbolic trig functions can be found in a similar fashion:

$$\boxed{\begin{aligned} \sinh^{-1} z &= \log\left(z + (z^2+1)^{1/2}\right) \\ \cosh^{-1} z &= \log\left(z + (z^2-1)^{1/2}\right) \\ \tanh^{-1} z &= \frac{1}{2}\log\frac{1+z}{1-z}. \end{aligned}}$$

Example As an illustration, we will find all solutions to the equation

$$\sin z = i.$$

The solutions are

$$-i\log\left(iz + (1-z^2)^{1/2}\right)$$

$$z = \sin^{-1} i = -i\log\left(i^2 + (1-i^2)^{1/2}\right)$$

$$= -i\log\left(-1 + 2^{1/2}\right)$$

$$= -i \log(-1 \pm \sqrt{z})$$

First look at $\log(-1+\sqrt{z}) = \ln|-1+\sqrt{z}| + i\arg(-1+\sqrt{z})$

$$= \ln|\sqrt{z}-1| + i\underbrace{2k}_{\text{even}}\pi \quad , k \in \mathbb{Z}.$$

Then look at $\log(-1-\sqrt{z}) = \ln|-1-\sqrt{z}| + i(2\ell+1)\pi \quad , \ell \in \mathbb{Z}.$

$$= \ln|1+\sqrt{z}| + i\underbrace{(2\ell+1)}_{\text{odd}}\pi$$

Then notice that

$$\ln|1+\sqrt{z}| = \ln\left|1+\sqrt{z}\,\frac{(-1+\sqrt{z})}{(-1+\sqrt{z})}\right|$$

$$= \ln\left|\frac{1}{\sqrt{z}-1}\right| = \ln|(\sqrt{z}-1)^{-1}|$$

$$= -\ln|\sqrt{z}-1|.$$

$$= \boxed{(-1)^{n+1} i \ln|\sqrt{z}-1| + \pi n \quad , n \in \mathbb{Z}}$$

//

Chapter 4: Integration

In this chapter, we develop the theory of integration of complex-valued functions of a complex-variable. Integrals will be defined over suitable curves in the complex plane. The theory of integration is a surprisingly powerful tool in the study of analytic functions.

The content of this chapter is essentially a characterization of analytic functions. Roughly speaking, we will prove the following theorem:

Let D be a domain and $f: D \longrightarrow \mathbb{C}$ a function. The following are equivalent:

(1) f is analytic on D;

(2) For all $n \in \mathbb{N}$, $f^{(n)}$ exists and is analytic on D;

(3) In each "simply connected" subdomain S of D, there is an analytic function $F: S \to \mathbb{C}$ such that $F' = f$ on S.

(4) f is continuous on D and

$$\int f(z)\, dz = 0$$

over every contour C lying in any "simply connected" subdomain.

(5) If C is a simple closed contour in D and z_0 is interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

//

Additionally, as an application of the theory, we will prove:

- Every bounded entire function is constant
- Every polynomial of degree $n \geq 1$ has atleast one root in $\mathbb{C}$.

//

Derivatives of Functions of a Real-Variable

To define integrals of a complex-valued function of a complex variable, we need to understand how to differentiate a function of a real-variable

$$w : I \subseteq \mathbb{R} \longrightarrow \mathbb{C}$$

where I is an interval in $\mathbb{R}$.

Definition Let $I \subseteq \mathbb{R}$ be an interval and $w : I \to \mathbb{R}$ a function. Writing $w(t) = u(t) + iv(t)$, we define the *derivative* of w to be

$$w'(t) = u'(t) + i\,v'(t)$$

provided that u' and v' exist. In that case, w is *differentiable*. //

Some rules for differentiation remain valid:

Proposition Suppose $w(t) = u(t) + i\,v(t)$ and $W(t) = U(t) + iV(t)$ are differentiable. Then

(1) $(w(t) + W(t))' = w'(t) + W'(t)$

(2) $(w(t)W(t))' = w'(t)W(t) + W'(t)w(t)$.

Note: there may be others.

Proof.

(1) $$(w(t) + W(t))' = (u + U + i(v + V))' = (u+U)' + i(v+V)'$$
$$= u' + U' + i(v' + V')$$
$$= (u' + iv') + (U' + iV')$$
$$= w'(t) + W'(t).$$

(2) $$(wW)' = ((u+iv)(U+iV))'$$
$$= (uU - vV + i(uV + vU))'$$
$$= (uU - vV)' + i(uV + vU)'$$
$$= u'U + U'u - (v'V + V'v) + i(u'V + V'u + v'U + U'v).$$

$$w'W = (u' + iv')(U + iV) = u'U - v'V + i(u'V + v'U)$$
$$W'w = (U' + iV')(u + iv) = U'u - V'v + i(U'v + V'u)$$

compare

■

Example We will frequently encounter the function

$$w(t) = e^{z_0 t}, \quad z_0 \in \mathbb{C}, \quad t \in [a,b].$$

We compute $w'(t)$ for use later:

Decompose w into real/imaginary parts: (write $z_0 = x_0 + iy_0$)

$$w(t) = e^{z_0 t} = e^{x_0 t + i t y_0}$$
$$= e^{x_0 t} \cdot e^{i t y_0}$$

$$= e^{x_0 t} \cos y_0 t + e^{x_0 t} i \sin y_0 t.$$

Then

$$w'(t) = x_0 e^{x_0 t} \cos y_0 t - \underline{y_0 e^{x_0 t} \sin y_0 t} + \underline{i}\left(x_0 e^{x_0 t} \sin y_0 t + \underline{y_0 e^{x_0 t}} \cos y_0 t \right)$$
$$= x_0 e^{x_0 t}(\cos y_0 t + i \sin y_0 t) + i y_0 e^{x_0 t} (\cos y_0 t + i \sin y_0 t)$$
$$= e^{x_0 t}(x_0 + i y_0)\, e^{i y_0 t}$$
$$= z_0 \, e^{x_0 t + i y_0 t} = z_0 e^{z_0 t}.$$

To summarize $\boxed{\dfrac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t}.}$

Integral of a Function $w : I \subseteq \mathbb{R} \to \mathbb{C}$

Definition (Definite Integral of $w : I \subseteq \mathbb{R} \to \mathbb{C}$) Suppose that $w(t) = u(t) + i v(t)$ where $u(t), v(t)$ are real-valued functions of a real variable defined on an interval $[a,b]$. The *definite integral* of w over $[a,b]$ is defined by

$$\int_a^b w(t)\, dt = \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt$$

provided that the integrals of u and v exist. //

provided that the integrals of u and v exist. //

Improper integrals over an unbounded interval are defined similarly.

$\boxed{\text{Example}}$ To illustrate the definition, we integrate $w(t) = e^{it}$ on $[0,\pi]$.

$$\int_0^{\pi} e^{it}\, dt = \int_0^{\pi} \cos t \, dt + i \int_0^{\pi} \sin t \, dt$$

$$= \quad 0 \quad + 2i$$

$$= 2i.$$

//

$\boxed{\text{Definition (Piecewise continuity)}}$ A function $u : I \subseteq \mathbb{R} \to \mathbb{R}$ is piecewise continuous on I if it is continuous on I except at a finite number of points at which it is discontinuous, but has one sided limits. A function $w(t) = u(t) + i v(t)$ is piecewise continuous if both u and v are. //

Note: The existence of the integrals

$$\int_a^b u(t)\, dt \quad \text{and} \quad \int_a^b v(t)\, dt$$

is ensured when w is piecewise continuous on $[a,b]$.

$\boxed{\text{Proposition (Properties of the integral of } w : [a,b] \to \mathbb{C})}$ Suppose that $w(t)$ and $W(t)$ are piecewise continuous on $[a,b]$. Then

(1) $\displaystyle\int_a^b z_0 w(t)\, dt = z_0 \int_a^b w(t)\, dt$ for any $z_0 \in \mathbb{C}$;

(2) $\displaystyle\int^b w(t) + W(t)\, dt = \int^b w(t)\, dt + \int^b W(t)\, dt$;

$$(2) \quad \int_a^b w(t) + W(t)\, dt = \int_a^b w(t)\, dt + \int_a^b W(t)\, dt \; ;$$

$$(3) \quad \int_a^b w(t)\, dt = \int_a^c w(t)\, dt + \int_c^b w(t)\, dt \, , \quad \text{any } c \in [a,b];$$

$$(4) \quad \int_a^b w(t)\, dt = - \int_b^a w(t)\, dt \, .$$

Proof. All follow from properties of ordinary integrals. ∎